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Original Research Article

GRAVITATIONAL DEFORMATION OF SOLID BODIES

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Abstract: Deformation of solid bodies under gravitation forces is of frequent occurrence and a subject matter of Mechanics. Usually calculations are performed for sphere-shaped solid bodies of a constant density. But if a sphere is compressed whilst its mass remains constant, a density is to increase in the process of deformation. Under these circumstances buckling and even collapse of a sphere are found to occur. Rotation of sphere-shaped solid bodies is also taken into consideration.

Keywords: elastic deformation; density increase; buckling; unconfined compression of solid bodies.

Introduction

Problems related to symmetrical deformation of the sphere with constant stuff density were considered in studies [1, 2]. They were tackled by the theory of elasticity methods provided that mechanical properties of the sphere stuff and its density were known. The problem factored in density variation was solved by another study [3].

If a solid sphere is compressed under the prohibition of mass loss, its density increases causing a structural strain of a body. The external pressure enhances gravitation, therefore gravitation and pressure deformation values cannot be summed up as it is customary to do in a linear elastic theory. Hence, there occurs a possibility of sphere buckling.

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1. Sphere Buckling

To consider buckling of the sphere subjected to gravitation and external pressure see Fig.1. A small body of $dr \times rd\varphi \times rd\varphi$ dimensions and mass

$$m_1 = \rho r^2 (d\varphi)^2 dr$$

is effected by radial gravitation force

$$F_r = \frac{\delta}{r^2} m_1 m_2$$

Where φ, r – are spherical system coordinates, m_2 – mass of sphere with radius r,
δ – gravitation constant.

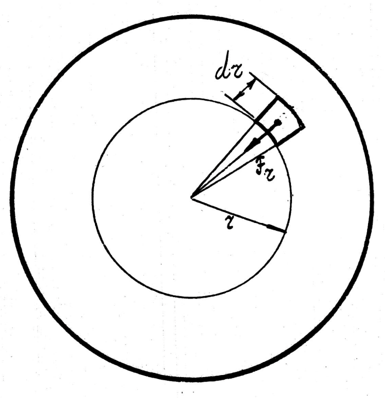

Fig.1. Scheme of Deformation of solid sphere

$$m_2 = 4\pi \int_0^r r_1^2 \rho(r_1) dr_1$$

where r_1 – additional argument, $\rho(r_1)$ – variable density under stuff compression.

By Newton's law the static equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) - \frac{4\pi\delta\rho(r)}{r^2}\int_0^r \rho(r_1) r_1^2 \, dr_1 = 0 \quad (1)$$

where σ_r, σ_θ – components of stress in a spherical coordinate system (the third main stress component σ_φ is equal to σ_θ).

Mean strain is determined by value

$$\varepsilon_0 = \varepsilon_r + 2\varepsilon_\theta = \frac{dU_r}{d_r} + \frac{2U_r}{dr} = \frac{1}{r^2} \times \frac{d}{dr}(U_r r^2)$$

where ε_0, ε_r are components of strain; u_r – radial displacement of body points under strain.

Density is equal to $\rho = \rho_0 \left[1 + \frac{1}{3}\varepsilon_0(r)\right]^{-1}$ and as it is shown in [3] it can change under deformation.

Given that density increases under compression but does not depend on coordinate r and sphere surface displacement is equal to $u_r(R) = -u_0 R$ (R – initial sphere radius without deformation, U_0 – non-dimensional value), the following formula can be developed:

$$\rho = \frac{\rho_0}{(1-U_0)^3} \quad (2)$$

where ρ_0 – initial stuff density; value u_0 describes a total change of sphere density.

Similar to [3], to solve the equation (1) is possible with the help of Hooke's law:

$$U_r = -br - \frac{ar(3-\gamma)}{2(1-\gamma)(1-U_0)^4} + \frac{ar^3(1+r)}{2R^2(1-\gamma)(1-U_0)^6} \quad (3)$$

$$\sigma_r = \frac{E}{1-2\gamma}\left\{-b - \frac{a(3-\gamma)}{2(1-\gamma)(1-U_0)^4}\left[1 - \frac{r^2}{R^2(1-U_0)^2}\right]\right\}$$

,

$$\sigma_\theta = \frac{E}{1-2\gamma}\left\{-b - \frac{a(3+\gamma)}{2(1-\gamma)^2(1-U_0)^4}\left[3 - \gamma - \frac{r^2(1+3\gamma)}{R^2(1-U_0)^2}\right]\right\} \quad (4)$$

where E – stuff elasticity modulus, γ – the Poisson ratio.

It is convenient to use two non-dimensional parameters: a and b

$$a = \frac{4\pi}{15E}\delta\rho_0^2 R^2(1-2\gamma) \quad (5)$$

$$b = \frac{\rho(1-2\gamma)}{E} \quad (6)$$

The first parameter (a) defines the relationship between gravitation and material rigidity, while the second parameter (b) – between external pressure P and material rigidity. For the surface of sphere with $U_r = -U_0 R$ and $r = R(1-U_0)$ it is possible to establish the following equation from (3):

$$U_0(1-u_0)^3 = a + b(1-U_0)^4 \quad (7)$$

2. Numerical Calculations

Fig.2 represents functions $U_0(a)$ when $b = 0$; 0.25; 0.50 and 1.0. If external pressure at the sphere surface is equal to zero, a lower curve is $U_0(a)$. At point A with $a = 0.105$, $U_0 = 0.25$ the derivative $\frac{dU_0}{da} \to \infty$ is infinite. Point A becomes an unstable point. The upper part of this curve cannot describe a real body deformation (it is represented by thin line in Fig.2).

If to assume that density remains constant and equal to ρ_0, the following equation can be obtained: $U_0 = a + b$. It is represented by the dashed line in Fig.2. It may be useful only if $U_0 \ll 1.0$ ($U_0 < 0.06$ see Fig.2). Under the very circumstances, gravitation and external pressure displacement values may be summed up.

So, there may be seen an upper limit of sphere dimension with radius *R*, under the given density ρ_0.

If $a_k = 0.105$, it is possible to derive from (5) a critical radius

$$R_k = \sqrt{\frac{15Ea_k}{4\pi\delta\rho_0^2(1-2\gamma)}}$$

$$R_k = \frac{0.354}{\rho_0}\sqrt{\frac{E}{\delta(1-2\gamma)}} \qquad (8)$$

Critical mass of a solid sphere corresponds to the value

$$m_k = \frac{4}{3}\pi R_k^3\rho_0 = \frac{0.186}{\rho_0^2}\left[\frac{E}{\delta(1-2\gamma)}\right]^{1.5} \qquad (9)$$

For example, if there is a solid stuff with

$E = 2\times10^{11}\frac{N}{m^2}$;

$\delta = 6.68\times10^{-11}\frac{m^3}{kg\times sec^2}$; $\gamma = 0.3$;

$\rho_0 = 7800\,{}^{kg}/{}_{m^3}$

the value of R_k makes

$$R_k = \frac{0.354}{7800}\sqrt{\frac{2\times10^{11}}{6.68\times10^{-11}\times0.4}} = 3.93\times10^6 m,$$

and $m_k = 2\times10^{24}kg$.

The equilibrium of such a sphere or larger in size spheres is unfeasible. When its density reaches 2.37 ρ_0, a gravitational collapse occurs.

Two assumptions are to be taken into consideration:

1. Hooke's law is valid and the elasticity modulus remains constant.
2. Newton's law of gravitation is an established truth.

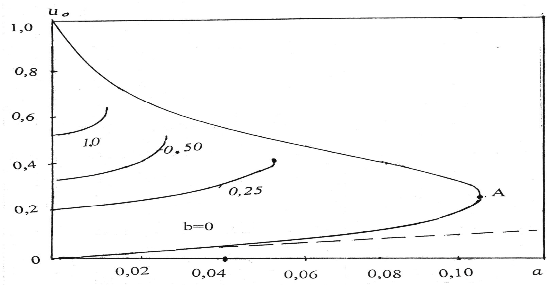

Fig.2. Diagram of functions $u_o(a)$

Reinforcement of elasticity modulus was considered in article [3]. If its change is not considerable, the main conclusions remain valid.

Plastic deformation was also discussed in [3]. It was proved that plastic deformation cannot invalidate the main conclusions [see 3, 4].

In case of gravitation collapse, when a gravity force is too strong the general theory of relativity laws are to be applied instead of the classical theory and Newton's law.

According to external-reference coordinate system when a collapse takes place and the radius approaches to a gravity radial value, all the body processes become slow [5].

$r_g = \frac{2\delta m}{c^2}$ where *C* is the velocity of light in vacuum.

In our example $m_k = 2\times10^{24}kg$;

$C = 3\times10^8\frac{m}{sec}$;

$$r_g = \frac{2\times6.68\times10^{-11}\times2\times10^{24}}{9\times10^{16}}$$

$= 2.97\times10^{-3}m$.

As may be inferred from the example given below, the relation between critical radius R_k and gravitational radius r_g is equal, so the Newton's law can be applied.

$\frac{R_k}{r_g} = \frac{3.93\times10^6}{2.97\times10^{-3}} = 1.3\times10^9$ and $R_k \gg r_g$.

Buckling and collapse will take place right before the necessity to apply the theory of relativity arises. It is well known that if a stellar mass is twice or thrice as much as the Sun mass, the gravitation collapse occurs and the star becomes a black hole [5]. Such process is eventual not only for plasma but for a solid body with definite characteristics.

3. Artificial Collapse Potential

The possibility of artificial buckling followed by a body collapse is very important nowadays. But this article does not aim at discussing the consequences of potential collapse in relation to the Earth (it is the subject of the other paper).

We would like to estimate the probability of such a phenomenon as itself. Based on equation (7), it is high. To produce a collapse and, hence, an artificial black hole it is sufficient to create the solid body surface pressure.

From equation (7) it can be inferred that if a surface of the sphere with $r = R(1-U_0)$ is subjected to external pressure with $b \neq 0$, then

$$\frac{dU_0}{da} \to \infty \text{ at } U_{0k} = \frac{1+4b}{4(b+1)} \qquad (10)$$

This value corresponds to a critical one:

$$a_k = \frac{27}{256(b+1)^3} \qquad (11)$$

Critical radius and critical mass are equal:

$$R_k = \frac{0.355}{\rho_{0(b+1)^{1.5}}}\sqrt{\frac{E}{\delta(1-2\gamma)}} = \frac{R_{ko}}{(b+1)^{1.5}},$$

$$m_k = \frac{m_{ko}}{(b+1)^{4.5}} \qquad (12)$$

where R_{ko} and m_{ko} are critical values of R_k and m_k, provided that external pressure is equal to zero and $P = b = 0$.

For example, if one wants to produce a collapse with the stuff mentioned above, namely: $E = 2\times10^{11}\frac{N}{m^2}$;

$\delta = 6.68\times10^{-11}\frac{m^3}{kg\times sec^2}$; $\gamma = 0.3$;

$\rho_0 = 7800\frac{kg}{m^3}$, but $m = 10\,000kg$ it is necessary to reduce m_k from $2\times10^{24}kg$ (when $P = 0$) by $\frac{2\times10^{24}}{10^4} = 2\times10^{20}$ times.

It is possible if $(b+1)^{4.5} = 2\times10^{20}$; $b = 3.21\times10^4 - 1$.

If $E = 2\times10^{11}\frac{N}{m^2}$; $\gamma = 0.3$ the applied pressure is calculated by formula (6):

$$P = \frac{E_b}{1-2\gamma} = \frac{2\times10^{11}\times3.21\times10^4}{0.4} = 1.6\times10^{16}\frac{N}{m^2}$$

In circumstances where the pressure $8.25\times10^4 E$ is very high, mechanical properties can vary considerably so one is to be careful in drawing conclusions. But now we cannot be absolutely certain that such a phenomenon as buckling of a solid sphere is impossible.

It is known that a solid sphere has natural oscillation frequencies.

$f_1 = \frac{2.37C_0}{R}$; $f_2 = \frac{6.1C_0}{R}$ …, where C_0 is the sound velocity, $C_0 = \sqrt{\frac{E}{\rho}}$

So, first two frequencies are $f_1 = \frac{2.37C_0}{R}\sqrt{\frac{E}{\rho}}$;

$$f_2 = \frac{6.1C_0}{R}\sqrt{\frac{E}{\rho}}.$$

For $m = 10^4kg$, $\rho_0 = 7800\frac{kg}{m^3}$; $E = 2\times10^{11}\frac{N}{m^2}$; $R = 0.674m$

$$f_1 = \frac{2.37}{0.674}\sqrt{\frac{2\times10^{11}}{7800}} = 1.78\times10^4 sec^{-1};$$

$$f_2 = 4.58\times10^4 sec^{-1}$$

But under compression the values R and ρ change, therefore resonance frequencies turn into off-resonance frequencies. Under compression R changes into $R(1-U_0)$ and ρ_0 into $\rho_0(1-U_0)^{-3}$ and

$$f_1 = \frac{2.37C_0}{R}\sqrt{\frac{E(1-U_0)}{\rho_0}} \qquad (13)$$

If one produces the external pressure oscillation on the sphere surface and changes its frequency by modifying $U_0(r)$ in accordance with equation (13), it becomes possible to incite resonance and collapse.

Sphere oscillation dynamics is intended to be treated at length elsewhere. Naturally, to do this it is necessary to carry out continuous measurements of displacement and value U_0.

The calculations of plastic deformation are considered in [3]. It is to take place on the sphere surface conditioned upon: $\sigma_r - \sigma_\theta = \sigma_s$ where σ_s is a yield stress of the stuff (see formula (4)). The calculations suggest that these are the very conditions for buckling phenomenon to occur. Change of volume and mean stress are proportional even in plastic deformation, therefore data on plasticity are compatible with main conclusions. The calculations for large deformations were performed with the help of Cauchy-Green strain tensor and tensor-gradient in [3].

Instead of equation (7) the calculation for John's semi-linear stuff suggests the following [3]:

$$U_0 \cdot (1 - U_0)^6 = a + b\,(1 - U_0)^8 \qquad (14)$$

The analysis of this equation shows that buckling occurs when

$$U_0 = 1 + \frac{7}{16b} - \frac{7}{16b}\sqrt{1 + \frac{192}{49}b}$$

and if $b \to 0$, then $U_0 \to \frac{1}{7}$

$$a_k = \frac{1}{7}\left(\frac{6}{7}\right)^6 = 0.057$$

Though critical value a_k differs from formula (7), the main conclusion of limiting values R_k and m_k is fully justified.

4. Rotation of Solid Sphere

If the sphere rotates about axis Z (see Fig.3) with angular velocity ω, every mass m_1 experiences not only a gravitational force but a centrifugal one:

$$F_c = m_1 \omega^2 r \times sin\varphi$$

This force acts straight across axis Z. Fig.3 shows the initial configuration of sphere with radius R as well as its shape deformed under the forces of gravitation and inertia. Under the circumstances the body is not a sphere anymore and its Z- axis displacement U_z is not equal to displacement in r-direction (Fig.3).

We shall call a sphere surface displacement in Z-direction – $U_1 R$ and a sphere surface displacement in the straight across axis Z direction – $U_2 R$.

Obviously, when a sphere rotates: $U_1 > U_2$ and even $U_2 < 0$, so configuration of a rotating sphere is close to ellipsoid with the volume:

$$V = \frac{4}{3}\pi R^3 (1 - U_1)(1 - U_2)^2.$$

Deformation of a rotating sphere with constant values ρ and ω was studied in [1]. But here we suppress previous assumption and the equations for the two displacement values U_1 and U_2 will be:

$$U_1(1-U_1)(1-U_2)^2 = b\,(1-U_1)^2(1-U_2)^2 + a + \frac{C\psi_1(1-U_1)}{(1-U_2)^2}$$

$$\left(U_2(1-U_1)(1-U_2)^2 = b\,(1-U_1)(1-U_2)^3 + a + \frac{C\psi_2(1-U_1)}{(1-U_2)^2}\right) \qquad (15)$$

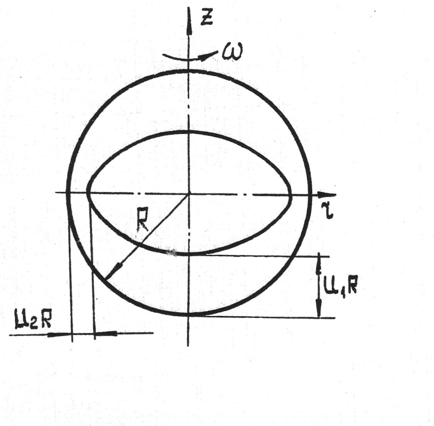

Fig.3. Scheme of rotating sphere

where a and b from equations (5, 6) go with additional non-dimensional parameter:

$$C = \frac{\rho_0}{E}\,\omega_0^2\, R^2 \qquad (16)$$

under initial values of density and angular velocity – ρ_0 and ω_0.

This parameter characterizes correlation between centrifugal forces and rigidity of sphere stuff.

Values ψ_1 and ψ_2 are functions of Poisson ratio γ.

$$\psi_1(\gamma) = \frac{2}{3}\frac{(2+\gamma)(1+\gamma)}{7+5\gamma} - \frac{2}{15}(1-2\gamma)$$

$$\psi_2(\gamma) = \frac{2}{15}(1-2\gamma) + \frac{(1+\gamma)(2+\gamma)}{3(7+5\gamma)} \qquad (17)$$

If $\omega_0 = 0$, $C = 0$, then $U_1 = U_2 = U_0$; instead of two equations (15) we get one (7).

But if $\omega_0 \neq 0$ and $C \neq 0$, it is necessary to take into account that body mass and its angular momentum remain constant. Therefore, the sphere density in compression increases while the moment of inertia J decreases. Angular velocity ω cannot remain constant as it has to increase in order to ensure the stability of angular momentum i.e. value $J\omega$. Thence we obtain:

$$\rho = \frac{\rho_0}{(1-U_1)(1-U_2)^2} \qquad (18)$$

$$\omega = \frac{\omega_0}{(1-U_2)^2} \qquad (19)$$

If there is neither loss of stuff nor action of external forces, a mass and an angular momentum are to remain constant. Momentum is equal to $J\omega = J_0\omega_0$ where J – moment of inertia and J_0 – its nominal value.

Fig.4 illustrates the calculation results of equation (15), for $\gamma = 0.3$; $\psi_1 = 0.18$; $\psi_2 = 0.17$ when $c = 0.1$ and $b = 0$.

We can see that if $a = a_k = 0.108$ ($c = 0.1$), buckling phenomenon takes place. If $a > 0.108$, equilibrium state is impossible. Judging from upper parts of the curves, compression in z-axis direction gives $U_1 = 1{,}0$ and zero thickness. But $U_2 \to 0.5$ and $c = 0.1$ induce a compression not to a point (black hole) but to a one-dimensional disk body with radius $R = 0.5$ and zero thickness. Fig.5 graphs $U_1(a)$ and $U_2(a)$ are made for $c = 10$ ($b = 0$). Buckling takes place when $a = a_k = 0.088$. In that case $U_1 \to 1{,}0$; $U_2 \to -0{,}30$ so there be formed a disk of zero thickness with R=1.3 ($U_2 = -0{,}30$). If $c = 10$, centrifugal forces are strong and collapse sphere diameter increases straight across axis Z (Fig.3).

Graphs of functions $U_1(a)$, $U_2(a)$ will correspond to those of Fig.6 if angular velocity increases, for instance up to $c = 50$. With the increase of ω_0 and c, the stability region of a solid body degrades. If $c = 50$, buckling and collapse occur when $a = 0.01$ $U_1 = 0.9$ $U_2 = -0.78$. In the vicinity of $a = 0.01$ a change is immediate and one can observe an acute angle in Fig.6a. To demonstrate a rapid

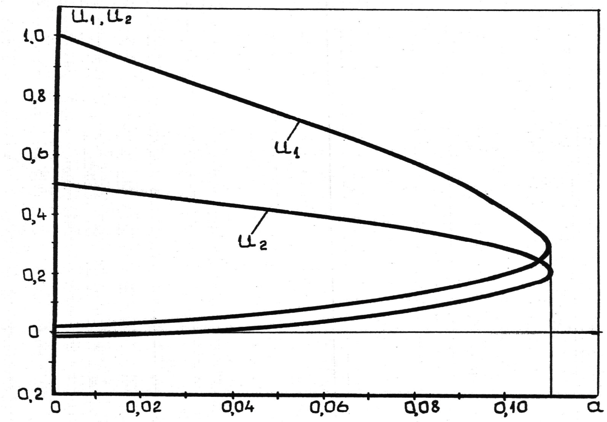

Fig.4. Graphs of functions $u_1(a)$,$u_2(a)$ by parameter c=0,1

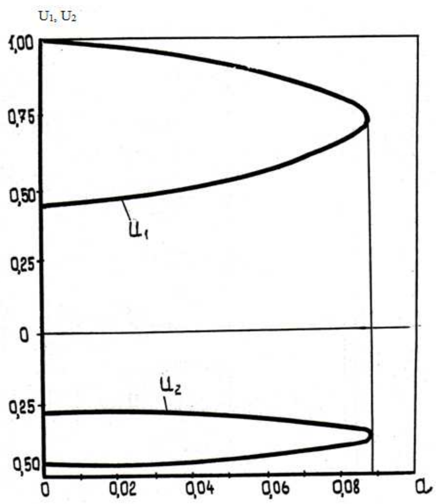

Fig.5. Graphs of functions $u_1(a)$,$u_2(a)$ by c=10

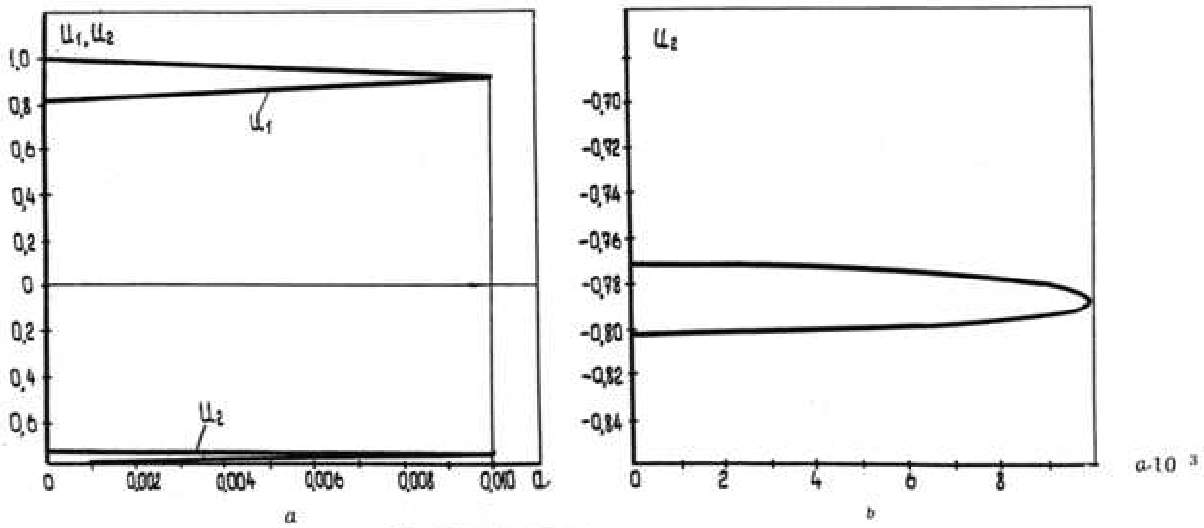

Fig.6. Graphs of functions $u_1(a)$,$u_2(a)$ by c=50

change of function $U_2(a)$ near value $a = 0.01$ a part of this curve near point $U_2 = 0.78$, $a = 0.01$ is scaled up (Fig.6b). Calculations suggest, when $c = 57$ there results a limiting case: the equilibrium is impossible and the stuff rigidity cannot resist its buckling. But there exists an angular velocity value $\omega_k = \frac{1}{R}\sqrt{\frac{57E}{\rho_0}} = \frac{7.55}{R}\sqrt{\frac{E}{\rho_0}}$ which is equal to $\omega_k = \frac{7.55}{10^3}\sqrt{\frac{2\times10^{11}}{7.8\times10^3}} = 38{,}2\, c^{-1}$ when sphere density $\rho_0 = 7.8 \times 10^3 \frac{kg}{m^3}$, radius $R = 1000m$ and modulus of elasticity $E = 2 \times 10^{11}\frac{N}{m^2}$.

Post-buckling equilibrium is impossible and it is a problem of dynamic deformation (it is the subject of the other paper).

But after the buckling takes place two variants are possible: rapture of sphere with its parts flying apart; and collapse with turning into the object of zero thickness and radius $1.8R$. Mass (m_0) and momentum $(J_0\omega_0)$, to be sure, remain constant. Such an object may be called 'black disk'.

Certainly, the first variant is more likely to occur but one cannot exclude the other alternative. Both variants of dynamic deformation result in explosion.

Conclusions

1. Existence of a sphere-shaped sold body of arbitrary large mass is impossible. There is a limiting value of mass when gravitation force produces buckling and unconfined compression to the point.
2. Solid body buckling can be produced artificially by pressure exerted on the sphere surface. The formulas in question are presented.
3. If a solid sphere rotates, its deformation under gravitation and inertia can result in buckling but without a central symmetry, which causes the transformation of a sphere into a disk of zero thickness and infinite density. Therefore, both the possible mass of a solid sphere and its permissible angular velocity of rotation are limited.

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